On the Coon amplitude.

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Zoom seminar series:
"Motives in QFT and string theory"

Based on F. Figueroa, PT, hep-th/2201.12331, PRL 2022

I. Review : Veneziano

II. Coon (2 → 2)

1. Unitarity
2. Low energy expansion

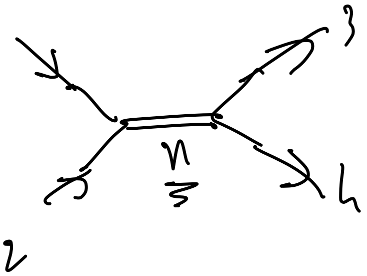
III. Generalizations

- N-points?
- other deformations
- KLT

I Veneziano

$$\boxed{A_V = \frac{\Gamma(-s+m^2)\,\Gamma(-t+m^2)}{\Gamma(-s-t+2m^2)} = B(-s+m^2, -t+m^2)}$$

$s, t, \ldots$ Mandelstam invariants

$\sum_{n=0}^{\infty}$ 

$p_i \quad i=1..4 \quad p_i^2 = m^2 \qquad -1 \leq m^2 \leq 1/3$

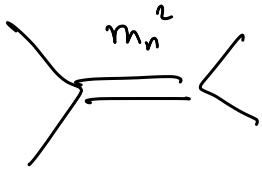
$(\alpha' = 1)$

Poles at $\quad s = m^2 + n = m_n^2 \quad n \geq 0$

$$\mathrm{Res}\, A_V(s,t)\Big|_{s=n+m^2} \propto \frac{\Gamma(-t+m^2)}{\Gamma(-n-t+m^2)} \propto (-t-1)\times(-t-2)\times\ldots\times(-t-n)$$

$\rightarrow$ Pol. of degree n in t

$$A_V(s,t)\Big|_{s\to} \sim \frac{P_n(t)}{s-m_n^2} \leftrightarrow$$ exchange of part with spins $= 0, 1, \ldots, n$



$$\boxed{A_V(s,t) = \sum_{n=0}^{\infty} \frac{P_n(t)}{s-m_n^2} = \sum_{n=0}^{\infty} \frac{P_n(s)}{t-m_n^2}}$$

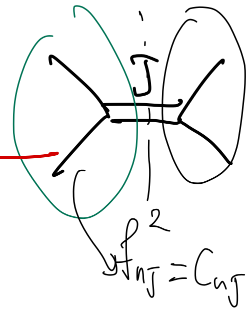
(need a few lines of computation)

$$P_n(t) = \sum_{J=0}^{n} c_{n,J}\, P_J\left(1 + \frac{2t}{m_n^2 - 4m^2}\right)$$

Legendre pols.

in d-dim Legendre → Gegenbauer $C_n^{(\alpha)}$ $\left(\alpha = \frac{d-3}{2}\right)$

① $\geq 0 \equiv$ UNITARITY

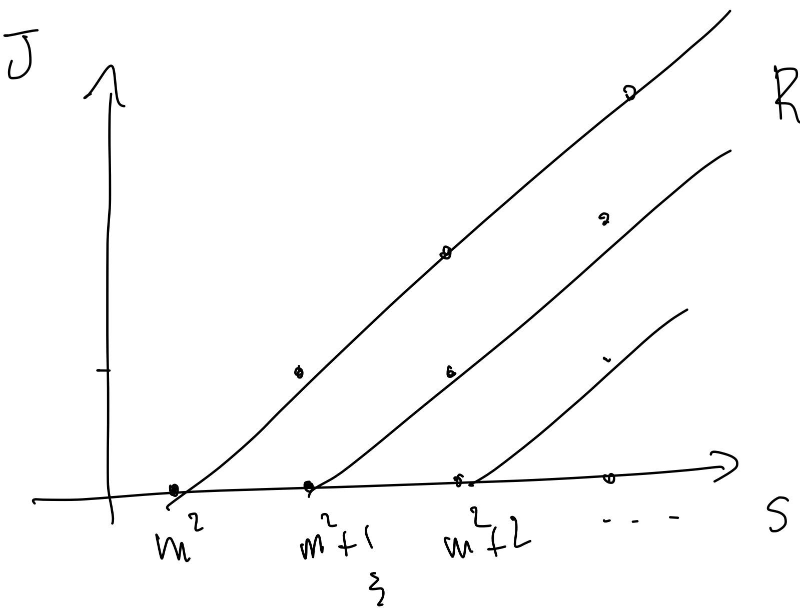


Bootstrap question:

Can one find $\{c_{n,J}, m_n^2\}$ s.t. ① & ② are true

② $$\sum_n \frac{\sum_J c_{nJ} P_J(\ldots t)}{s - m_n^2} = \sum_n \frac{\sum_J c_{nJ} P_J(\ldots s)}{t - m_n^2}$$

Only known solution* = Veneziano (because can not exactly meromorphic)



Regge trajectories

$2 \to 2$

$n \to m$

$$\int_{0<z_1 \ldots <z_n<0} dz_1 \ldots dz_{n-3}\ (K-N)$$

Recall

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x \; n!}{x(x+1)\cdots(x+n)}$$

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = \prod_{n=0}^{\infty} \frac{(x+y+n)\; n!}{(x+n)(y+n)}$$

$x = -s + m^2$

black: original Coon ansatz
red: generalisation, F. Figueroa, PT unpublished

Coon '69

Ansatz:
$$A(s,t) = \prod \frac{\left(1 + (s+t) f_n + st\, g_n + (s^2 + t^2) + \dots + s^2 t^2\right)}{\left(1 - s/p_{2n}\right)\left(1 - t/p_{2n}\right)\left(1 - s/p_{2n+1}\right)\left(1 - t/p_{2n+1}\right)}$$

$\exists ?\; p_n,\ f_n,\ g_n$

s.t. $\operatorname{Res} A \big|_{s=p_n} \quad \tilde{P}_n(t)$

$s = p_k$

$$\prod_{n=0}^{\infty} \frac{\left(1 + (p_k + t) f_n + p_k \cdot t\, g_n\right)}{\left(1 - p_k/p_n\right)\left(1 - t/p_n\right)}$$

⚠ $p_n \equiv$ poles
$P_n(t) \equiv$ residue

$\rightarrow t = p_n$

$$\left(\tilde{P}_n(t)\right) = \frac{\prod_{n=0}^{\infty} \left(1 + (p_k + t) f_{n+k} + p_k \cdot t\, g_{n+k}\right)}{\prod_{n=0}^{\infty} \left(1 - t/p_n\right)}$$

Want to find p, f, g st

$\forall k, n \geq 0 \quad 1 + (p_k + p_n)\, \cancel{f_{n+k}} + p_k \cdot p_n\, \cancel{g_{n+k}} = 0$

See Coon's original work for this derivation

$$D_k(p(k)) = D_n(p(n)) = c.$$

Schwarzian differential.

Find $\boxed{p_k = m^2 + \frac{q^k - 1}{q - 1}} \longrightarrow \underline{f_n} \quad g_n$

$\sigma = 1 + (s - m^2)(q-1)$

$\tau = 1 + (t - m^2)(q-1)$

$$A_c(s,t) = (q-1) \prod \frac{(\sigma\tau - q^n)(1 - q^{n+\frac{1}{2}})}{(\sigma - q^n)(\tau - q^n)}$$

$\sigma\tau \to (s+t) \quad s \times t$

$\sigma = q^n$

$\to \quad s = m^2 + \frac{q^n - 1}{\cancel{q-1}} \xrightarrow[q \to 1]{} m^2 + n$

$\to \cancel{(q-1)}\, n$

$= m^2 + [n]$

What about q?

NOT NICE

$q > 1 \longrightarrow A_c(s,t)$ has polynomial res. NON UNITARY

$\rightarrow$ $q < 1$

$$A_c(s,t) = \prod \left[\frac{(\sigma\tau - q^n)(1 - q^{n+1})}{(\sigma - q^n)(\tau - q^n)} \right]$$

$$= \prod \frac{(1 - q^n/\sigma\tau)(1 - q^{n+1})}{\cancel{(1 - q^n/\sigma)}(1 - q^n/\tau)}$$

$$\mathrm{Res}\,\Big|_{\sigma = q^k} \propto \prod_{n=0}^{\infty} \frac{(1 - q^{n-k}/\tau)}{(1 - q^n/\tau)} = \left(1 - \frac{q^{-k}}{\tau}\right) \times \dots \left(1 - \frac{q^{-1}}{\tau}\right)$$

$$\times \frac{\prod_{n=k}^{\infty} (1 - q^{n-k}/\tau)}{\prod_{n=0} (1 - q^n/\tau)}$$

$$\tau = (t - t_*)(\dots)$$

$$\mathrm{Res}\, A_c = P_n \left(\frac{1}{t - t_*} \right)$$

$$\boxed{\widehat{A}_c(s,t) = q^{\frac{\ln \sigma}{\ln q}\frac{\ln \tau}{\ln q}} \prod \frac{\left(1 - q^n/\sigma\tau\right)\left(1-q^{n+1}\right)}{\left(1-q^n/\sigma\right)\left(1-q^n/\tau\right)}}$$

$\sigma = q^n$

$0 \leq q \leq 1$

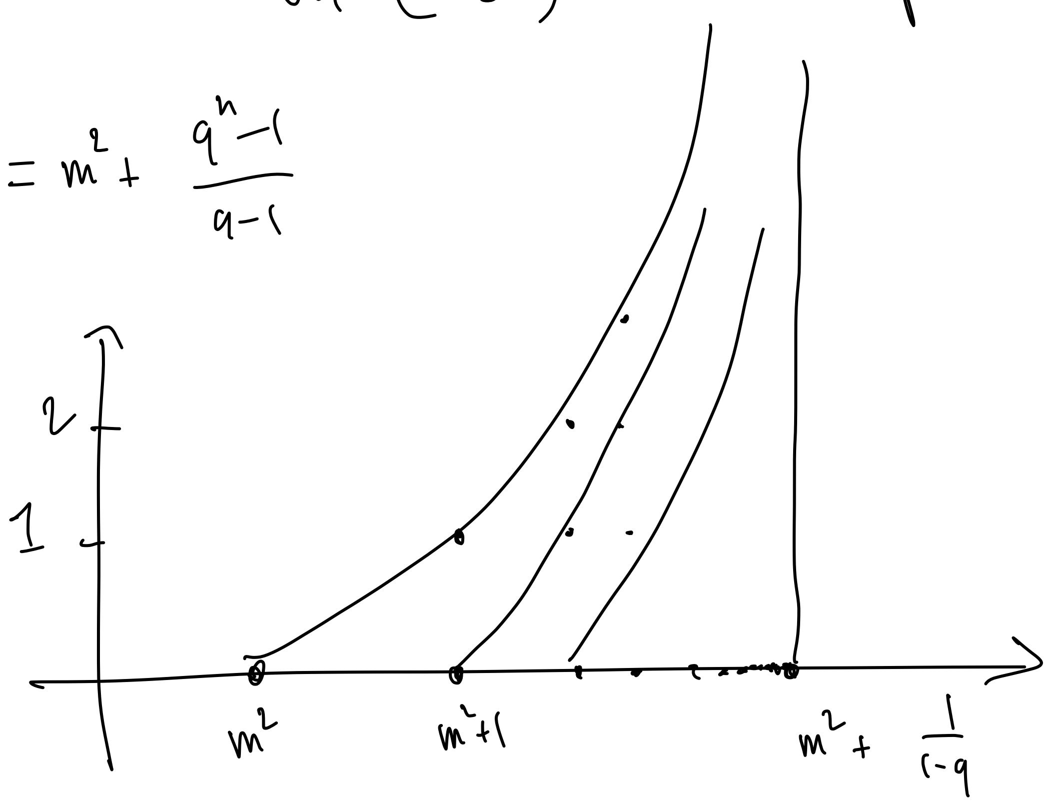
$$e^{n.\ln\tau} = \tau^n$$

$$\mathrm{Res} \rightsquigarrow \mathrm{Res} \propto \tau^n \tilde{P}_n\left(\frac{1}{\tau}\right) = \tilde{\tilde{P}}_n(\tau)$$

$$q^{\ln(1+(s-m^2)(q-1))}$$

$\ln(s) \sim$ loops.

$$s = m^2 + \frac{q^n - 1}{q-1}$$



Low energy expansion

$m^2 = 0 \qquad s+t+u = 0 \qquad a \doteq 1 \qquad \boxed{\alpha' - \text{expansion}}$

$$A_c(s,t,u) = A_c(s,t) + A_c(t,u) + A_c(u,s)$$

$$= \frac{1}{\alpha'}\left(\frac{1}{s}+\frac{1}{t}+\frac{1}{u}+\right) c_0(q) + \alpha'^2 c_2(q)(s^2+t^2+u^2) + c_{3(q)}\alpha'^3(stu)$$

$+ \dots$

$c_0(q) = (1-q)$

$$2 \leftarrow c_2(q) = \frac{1}{2}(q-1)^3\Big(3\underbrace{h_1(q)}_{(q-1)} + 5\underbrace{h_2(q)}_{(q-1)^2} + 2\underbrace{h_3(q)}_{(q-1)^3}\Big) - \frac{(q-1)^3}{\log q}$$

$$h_m(q) = Li_m(q^m, q) = \sum_{n=1}^{\infty} \frac{q^{nm}}{\left(\frac{1-q^n}{1-q}\right)^m}$$

$q^{nm} \to 1$

$\left(\frac{1-q^n}{1-q}\right)^m \to [n]^m \to n^m$

Schlesinger math/0111022

$\to \zeta_m$

$(q-1) \to$ has transc. wg. $\longrightarrow$ Geiser, Lindwasser

$$A_c = \frac{\Gamma_q(-\ln_q \sigma)\ \Gamma_q(-\ln_q \tau)}{\Gamma_q(-\ln_q(\sigma\tau))} = B_q(\ldots)$$

Askey '80

$$\Gamma(1+z) = \exp\left(-\gamma_\epsilon z + \sum_k \frac{\zeta_k}{k}(-z)^k\right)$$

Conclusion

- N-point Fct? Not known.

$q>1$

→ Selberg w/ Yihong-Wang, FF, PT

↓

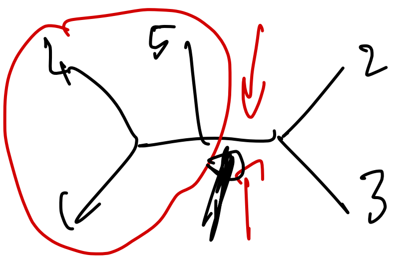
$\sum t \cdot \binom{\ }{\ }$

$$\boxed{\int [dt]_q\, t^x (1-t)_q^y = B_q(x,y)} \qquad Li_n(\dots,q)$$

$$\int [dt_1]_q [dt_2]_q (\dots)$$

5-point Coon. → $\boxed{\frac{\Gamma_q\,\Gamma_q\,\Gamma_q\,\Gamma_q}{\Gamma_q\,\Gamma_q}\; {}_3\phi_2(\dots)}$

Factorisation is a problem



$\times$

G_{ij} $\quad s_{ij}$

$s_{14}+s_{45}+s_{51} = \# = s_{23}$

s_{45} $\quad s_{51}$

$$\boxed{G_{45}\,G_{51}\,G_{14} = q^n} \to \left(1+(s_{45}-m^2)(q-1)\right)\times\dots = q^n$$

$$\left((s_{14}-m^2)+(s_{45}-m^2)+(s_{51}-m^2)\right)(q-1) + (\dots) = q^n-1$$

$\boxed{(s_{ij})^2 \quad (s_{ij})^3}$

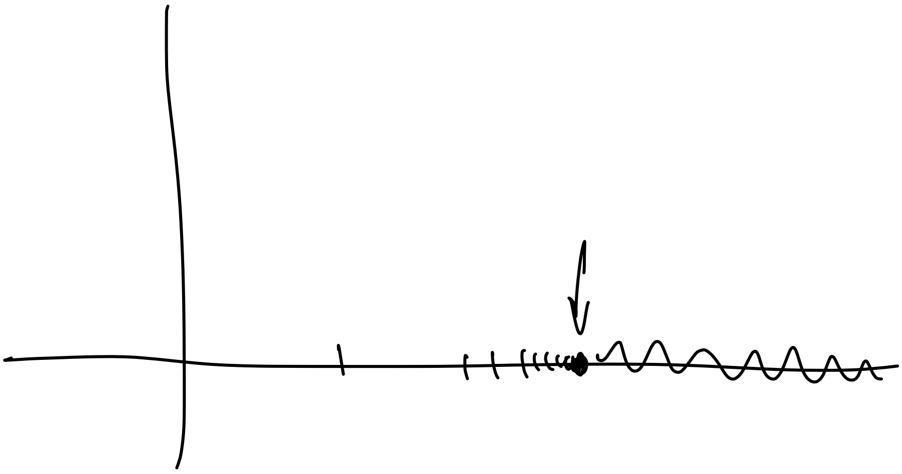
KLT. Coon $\to$ NO.

$$\left(\frac{\Gamma_q(s)\,\Gamma_q(t)}{\Gamma_q(s+t)} \right) \text{ has KLT}$$

$\to$ q^s q^t Res $\to$ $P_n(q^t)$

In F. Figueroa, PT (unpublished),
twisted period relations for q-Beta function.

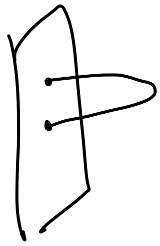
physical interpretation.



$$E_n = \frac{1}{n^2}$$

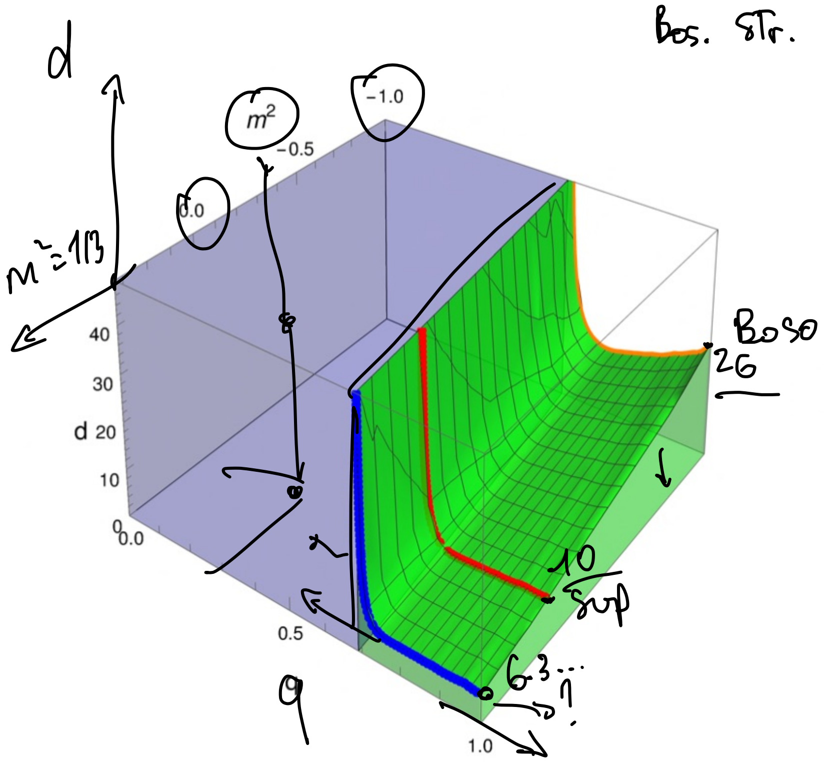
(Klebanov. Maldacena

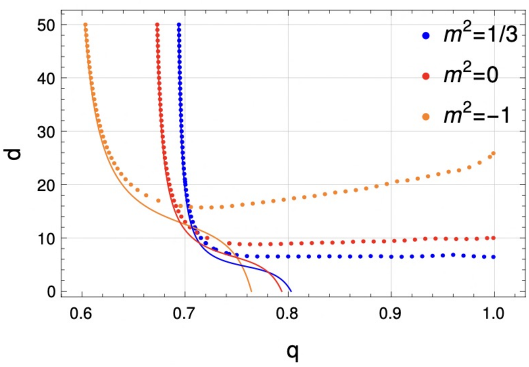
Maldacena Remmen)



$$E_n \sim e^{-n..}$$

Bos. STr. $m^2=-1$ $\quad d=26$





in which dimension d are all $c_{n,J} \geq 0$